Individual Accountability, Collective Decision-making: Appendix

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1 Overview

In this Appendix, I consider a general version of the model in which voters learn the state after the group selects policy and before the election with probability $\rho \geq 0$. The baseline model I analyze in the main text assumes $\rho = 0$.

In Section 2 I formally define beliefs and strategies for legislators. In Section 3 I analyze a SPIE. Section 3.1 provides a complete formal definition of a SPIE. Section 3.2 describes legislator beliefs in a PIE. In Section 3.3 I characterize the ex ante probability that a group of low-ability members selects $y = 0$ in a PIE for all q and n. Section 3.4 uses this result to characterize voter beliefs in a PIE for all q and n. These properties of the PIE are stated formally as Lemmas. I provide proofs for these in Section 5. Section 3.5 provides a Lemma that establishes necessary and sufficient conditions for a SPIE to exist for $\rho \geq 0$. Proof of this Lemma is also in Section 5. Section 4 provides proofs for all claims in the main text using the Lemmas from Sections 3.3-3.5.

2 Legislator Strategies and Beliefs

Let $m \in (\Theta \times S)^n$ denote the n-tuple of messages that legislators send in the communication stage. Let m_{-i} denote the $n-1$ messages from his colleagues that legislator i observes. A legislator's belief about the state of the world is given by

$$\zeta_i(\theta_i, s_i, m) = Pr(\omega = 0|\theta_i, s_i, m)$$

Legislators' beliefs about the state of the world are derived from their beliefs about the type of all legislators. Let $\Psi \subset (\Theta \times S)^n$ denote the set of all possible type realizations and let ψ represent a member of this set.[1] Each member's beliefs given m and his private information are represented by a probability measure on Ψ, η_i. It will be useful in defining and analyzing

[1] Formally, $\Psi = (\Theta \times S \setminus (H, 1))^n \cup (\Theta \times S \setminus (H, 0))^n$.

equilibrium to let $\Psi^c = (\Theta \times S)^n \setminus \Psi$ be the set of all type permutations that cannot be realized. This is the set of all n-tuples for which there is at least one member who reports that he is type $(H, 1)$ and at least one who reports that he is type $(H, 0)$. It will also be useful to let ψ_i denote the i-th element of ψ and ψ_{-i} refer to the $n-1$ elements of ψ other than i.

A pure strategy for a legislator consists of a voting strategy and a messaging strategy. A voting strategy is a mapping from a legislator's type, (θ_i, s_i), and the messages sent in the communication stage of the game, m, into the policy space:

$$\tilde{v}_i : (\Theta \times S)^{n+1} \to \{0, 1\}$$

where $\tilde{v}_i(\theta_i, s_i, m) = 0$ denotes that legislator i of type (θ_i, s_i) votes for policy 0 if the set of messages sent is m. A messaging strategy is a mapping from a legislator's type into the individual type space,

$$\tilde{m}_i : \Theta \times S \to \Theta \times S$$

For example, $\tilde{m}_i(L, 0) = (L, 1)$ denotes that member i sends the message $m_i = (L, 1)$ if he is of type $(L, 0)$.

3 Public Interest Equilibrium

3.1 Formal Definition

Below is the complete formal definition of a PIE.

Definition 1 (Sincere Public Interest Equilibrium) *The following is a sincere public interest equilibrium:*

Legislators truthfully report their type:

$$\tilde{m}_i^*(\theta_i, s_i) = (\theta_i, s_i) \quad \textit{for all} \quad i$$

Legislators vote for the policy they believe is most likely to match the state:

$$\tilde{v}_i^*(\theta_i, s_i, m) = 0 \quad \textit{if and only if} \quad \zeta_i(\theta_i, s_i, m) \geq 1/2 \quad \textit{for all} \quad i$$

Voter beliefs satisfy Bayes' rule.

For any $m \in \Psi$,

$$\eta_i(\psi) = \begin{cases} 1 & \textit{if} \quad \psi_{-i} = m_{-i} \quad \textit{and} \quad \psi_i = (\theta_i, m_i) \\ 0 & \textit{otherwise} \end{cases}$$

For any $m \in \Psi^c$, $\eta_i(\psi) > 0$ if and only if $\psi_j = (H, 1)$ for all legislators with $m_j = (H, 1)$.

3.2 Legislator Beliefs in PIE

In the communication stage of the policy-making process, any message that corresponds to a possible realization of n types occurs with positive probability in equilibrium. On the equilibrium path, all legislators are truthful and Bayes' rule implies that all legislators believe that $\psi = m$ with probability one. If a legislator lies, the resulting set of messages may still be one that is achieved with positive probability in equilibrium. By Bayes' rule, any legislator who does not lie believes that $\psi = m$ with probability one. Legislators who break from equilibrium in the messaging stage have free beliefs in the voting stage. In principle the lying legislator could believe one or more of his colleagues is also lying or even believe that he is of a different type than he believed prior to deviating from equilibrium. I assume that a deviant legislator believes his fellow legislators are telling the truth and that he is of the type that he observed at the start of the game.

A lie in the communication stage may also result in a message that cannot represent the true realization of n types—i.e. $m \in \Psi^c$. In this case at least one legislator sends $m_i = (H, 1)$ and at least one legislator sends $m_j = (H, 0)$. I assume that each legislator believes that the legislator who sends the message $m_i = (H, 1)$ is of type $(H, 1)$ with probability one.

Only these beliefs satisfy a strict dominance refinement of weak sequential equilibrium. All members are strictly worse off relative to a PIE if $y = 1$ is selected with a strictly higher probability than in a PIE. If any type lies that he is type $(H, 1)$ and all other members are low ability, m is a message that is reached with positive the equilibrium path and the other members believe that $y = 1$. The lie therefore raises the probability that the group incorrectly selects $y = 1$ in the event that all members are low ability. This event occurs with strictly positive probability. If $\omega = 1$ and at least one other member is high ability, then m is on the equilibrium path. His lie is not detected and appears as a redundant piece of evidence that $\omega = 1$. The lie has no effect on the group's decision in this event. If $\omega = 0$ and at least one other member is high ability, then $m \in \Psi^c$. In this event, the deviant member's highest payoff occurs if the members select $y = 0$. This is the same probability as in a PIE. Therefore even in the best possible case for how legislators act at off-path information sets, misreporting $m_i = (H, 1)$ strictly raises the probability that $y = 1$ relative to a PIE.

3.3 Low-Ability Group Behavior in PIE

In a PIE, a group with at least one high-ability member chooses $y = 0$ with ex ante probability π. Let

$$\lambda(q) \equiv Pr(y = 0 | \theta_i = L \, \forall \, i)$$

denote the ex ante probability that a group of low ability members selects $y = 0$. I refer to a group of low types as *uninformed* if $q \leq \pi$ and *informed* if $q > \pi$. If $q = 1/2$, low types are *completely uninformed.* If $q = 1$, low types are *completely informed.* If $q = \pi + \varepsilon$ for ε arbitrarily small, low types are *barely informed.* If $q = \pi$, low types are *barely uninformed.*

To characterize $\lambda(q)$, it is useful first to identify how the group makes its decision. In a PIE all information is shared. The group selects the policy that is most likely to be correct by applying Bayes' rule to the realization of signals it receives. Define n_0 as the number of $s_i = 0$ signals in the group of n low ability members. For all $q \in [1/2, 1]$ there is a unique minimum number of $s_i = 0$ signals, $n_0(q)$, such that $Pr(y = \omega | n_0) \geq 1/2$ if and only if $n_0 \geq n_0(q)$. Thus in a PIE, $n_0(q)$ is the number of $s_i = 0$ signals such that the group chooses $y = 0$ if $n_0 \geq n_0(q)$ and $y = 1$ if $n_0 < n_0(q)$. Let $N_0 \equiv \{0, 1, 2, ..., \frac{n-1}{2}, \frac{n+1}{2}\}$. For a given n, N_0 is the set of all non-negative integers less than or equal to a simple majority of n. With this notation, $n_0(q)$ is mapping from $[1/2, 1]$ to N_0.

Lemma 1

$n_0(q)$ is increasing in q

$n_0(q) = \frac{n+1}{2}$ *for all* $q > \pi$

$n_0(\pi) = \frac{n-1}{2}$

$n_0(1/2) = 0$

$n_0(q)$ is onto N_0

The first part of Lemma 1 states that as the precision of low ability members' signals increases, the minimum proportion of $s_i = 0$ signals for which the group selects $y = 0$ rises. The second part states that groups of informed low ability types choose $y = 0$ if and only if a simple majority of member receive a $s_i = 0$ signal. The third part states that a group of barely uninformed low ability members chooses $y = 0$ unless a simple majority plus at least one other member receives a $s_i = 1$ signal. The fourth part states that a group of completely uninformed low-ability members always chooses $y = 0$. The final part of Lemma 1 states that for all $n_0 \in N_0$, $n_0(q) = n_0$ for some q.

The probability that a group of low types chooses $y = 0$ can be expressed as

$$\lambda(q) = \pi(1 - F(n_0(q) - 1, n, q)) + (1 - \pi)(1 - F(n_0(q) - 1, n, 1 - q)) \tag{1}$$

where F is the cdf of the binomial distribution. That is, $F(n_0(q) - 1, n, q)$ reports the probability of less than or equal to $(n_0(q) - 1)$ successes (where a success is a $s_i = 0$ signal) out of n trials with success probability q. Note that q affects $\lambda(q)$ through two channels: i) the minimum number of $s_i = 0$ signals such that the group selects $y = 0$ and ii) the state-conditional probability that each member receives a $s_i = 0$ signal. Lemma 2 identifies properties of $\lambda(q)$ as a function of q.

Lemma 2

$\lambda(q)$ *is strictly increasing and continuous on* $(\pi, 1]$ *with* $\lambda(1) = \pi$

$\lambda(q)$ *is decreasing on* $[1/2, \pi]$ *with* $\lambda(1/2) = 1$ *and* $\lambda(\pi) > \pi$

$\lambda(\pi) - \pi > \pi - \lim_{q \to \pi^+} \lambda(q)$

The first part of Lemma 2 states that a group of informed low types chooses $y = 0$ with a lower probability than a group with at least one high type. As the precision of signals for informed low types increases, a group of informed low types chooses $y = 0$ with a strictly and continuously higher probability. A group of completely informed low types chooses $y = 0$ with the same probability as a group with at least one type. The second part of Lemma 2 states that a group of uninformed low types chooses $y = 0$ with a strictly higher probability than a group with at least one high type. A group of completely uninformed low types choose $y = 0$ with probability one. As the precision of signals for uninformed low types increases, the probability with which a group of uninformed low types chooses $y = 0$ is (weakly) decreasing. The final part of Lemma 2 states that a group of barely informed low types chooses $y = 0$ with probability closer to π than a group of barely uninformed low types.

Note from (1) that $\lambda(q)$ is also influenced by n. As n rises, more information is available to the group in a PIE and the group selects the correct policy with a higher probability. Because $y = 0$ is more likely to be the correct policy, a group of low-ability members selects $y = 0$ with a higher probability as n rises. From Lemma 2, this implies that as a group of informed low ability members grows, it selects $y = 0$ with a weakly higher probability. As a group of uninformed low ability members grows, it selects $y = 0$ with a weakly lower probability. Lemma 3 states this formally.

Lemma 3

$\lambda(q)$ is decreasing in n for all $q \leq \pi$.

$\lambda(q)$ is increasing in n for all $q > \pi$.

3.4 Voter Beliefs in PIE

A voter's beliefs about her representative's ability are derived from her beliefs over the set of all possible ability and signal realizations which are represented by a probability measure on Ψ. To economize on notation, I characterize voter beliefs only up to beliefs about their representative's type.

Let I denote a random variable that equals ω with probability ρ and $\emptyset$ with probability $1-\rho$. An information set for the voters can be expressed now as a pair (y, I). Let Φ denote the set of six information sets and let ϕ denote an element of this set. At each possible information set, each voter forms beliefs about the probability that her representative is a high ability type, $\mu_i(\phi)$.

It will be useful to identify several properties of voter beliefs when voters do not observe ω. In a PIE,

$$\mu_i(0, \emptyset) = \frac{\pi}{\pi + (1 - \frac{1}{2^{n-1}})\pi + \frac{1}{2^{n-1}}\lambda(q)} \tag{2}$$

$$\mu_i(1, \emptyset) = \frac{1-\pi}{1-\pi + (1 - \frac{1}{2^{n-1}})(1-\pi) + \frac{1}{2^{n-1}}(1-\lambda(q))} \tag{3}$$

Lemma 4 (Properties of voter beliefs in q)

- *If $q > \pi$*

  $\mu_i(1, \emptyset) \leq 1/2 \leq \mu_i(0, \emptyset)$

  $\mu_i(1, \emptyset)$ *is strictly increasing and continuous in q*

  $\mu_i(0, \emptyset)$ *is strictly decreasing and continuous in q*

  $\mu_i(1, \emptyset) = \mu_i(0, \emptyset) = 1/2$ *if $q = 1$*

- *If* $q \leq \pi$

  $\mu_i(0, \emptyset) < 1/2 < \mu_i(1, \emptyset)$

  $\mu_i(1, \emptyset)$ *is decreasing in* q

  $\mu_i(0, \emptyset)$ *is increasing in* q

- $|\mu_i(1, \emptyset) - \mu_i(0, \emptyset)|$ *is decreasing in* q

Lemma 5 (Properties of voter beliefs in n**)** *If* $q > \pi$, $\mu_i(0, \emptyset)$ *is decreasing in* n *and* $\mu_i(1, \emptyset)$ *increasing in* n*. If* $q \leq \pi$, $\mu_i(0, \emptyset)$ *is increasing in* n *and* $\mu_i(1, \emptyset)$ *decreasing in* n*.*

3.5 Existence of PIE

Lemma 6 provides necessary and sufficient conditions for the existence of a PIE for the general model in which $\rho \geq 0$. Proposition 1 then follows from Lemma 6 for $\rho = 0$.

Lemma 6 *If* $q \in (\pi, 1)$*, a public interest equilibrium exists if and only if at least one of the following conditions is true:*

i) No legislator is in a close race: $k_i \notin (\mu_i(1, \emptyset), \mu_i(0, \emptyset)]$ *for all* i*.*

ii) The probability of state revelation is sufficiently high:

$$\rho \geq \bar{\rho} \equiv \frac{\pi(1-q) + (1-\pi)q}{2(1-\pi)q}$$

4 Proofs of Claims in Main Text

Proposition 1 *A sincere public interest equilibrium exists if and only if no member is in a close race.*

Proof of Proposition 1:

I assume in the baseline model in the main text that $\rho = 0$. Therefore $\rho < 1/2 < \bar{\rho}$. Lemma 6 then implies the proposition. □

Proposition 2 *A PIE exists if and only if a SPIE exists.*

Proof: In the main text I establish that a SPIE is a PIE. Trivially then, existence of a SPIE implies that a PIE exists.

To show necessity, suppose a SPIE does not exist. Define

$$y^*(\psi) \equiv \underset{y\in\{0,1\}}{\text{argmax}} \ (1-y)Pr(\omega = 0|\psi) + yPr(\omega = 1|\psi)$$

By definition, an equilibrium is a PIE if and only if for all ψ, the group selects $y \in y^*(\psi)$ with probability one. For $q > \pi > 1/2$, $y^*(\psi)$ is a singleton for all ψ. To see this, note that for all ψ with at least one high ability member, either $Pr(\omega = 0|\psi) = 1$ or $Pr(\omega = 1|\psi) = 1$. For ψ with all low ability member, $q > \pi > 1/2$ imply that $Pr(\omega = 0|\psi) > 1/2$ for $n_0 \geq \frac{n+1}{2}$ and $Pr(\omega = 1|\psi) > 1/2$ for all $n_0 \leq \frac{n-1}{2}$. Thus in every PIE, $y = y^*(\psi)$ with probability one for all ψ. It follows that for each y and θ_i, $Pr(y|\theta_i)$ is the same in every PIE. Thus in every PIE, voter posterior beliefs $\mu_i(y)$ are the same as in a SPIE.

Non-existence of a SPIE implies that there is some member i in a close race. Equivalence of $\mu_i(y)$ in all PIE imply that member i is in a close race in every PIE. Thus member i strictly prefers the group to select $y = 0$ for all ψ. A profitable deviation for member i exists if there is an alternative strategy he can play that raises the probability that the group selects $y = 0$. In particular, if an alternative strategy induces the group to select $y = 0$ for all ψ such that $y^*(\psi) = 0$ and at least one ψ for which $y^*(\psi) = 1$, a profitable deviation exists.

Consider the following alternative strategy. Member i plays the strategy that equilibrium prescribes if he is type $(H, 0)$, $(L, 0)$, or $(H, 1)$. If he is type $(L, 1)$, he plays the strategy prescribed by equilibrium if is a $(L, 0)$ type. Because type is private information and he is type $(L, 0)$ with positive probability for both states, the deviant strategy yields a set of messages, m, that is reached with positive probability in equilibrium. PIE requires that if all members are low ability and exactly half of the other $n-1$ members receive each signal, $y^*(\psi) = 0$ if and only if member i's type is $(L, 0)$. The deviation therefore changes the group's decision from $y = 1$ to $y = 0$ if and only if all members are low and $n_0 = \frac{n-1}{2}$. The deviation is therefore profitable for member i. □

Proposition 3 *In larger groups of legislators, fewer individual races are close.*

Proof of Proposition 3: Immediately implied by Lemma 5. □

To prove Corollary 1 and Proposition 4, it will be helpful to let $\mu_i^n(y, \emptyset)$ denote a voter's posterior belief in a PIE in an n-member group.

Corollary 1 *Fewer individual races are close for any member of a group than for a single decision maker.*

Proof of Corollary 1:

A single decision-maker serves the public interest if and only if he selects $y = s_i$ in equilibrium. In such an equilibrium, $\mu_i^1(0, \emptyset) = \frac{\pi}{\pi + \pi q + (1-\pi)(1-q)}$. Note that

$$\pi q + (1-\pi)(1-q) = (1 - \frac{1}{2^{n-1}})\pi + \frac{1}{2^{n-1}}[\pi F(\frac{n+1}{2}, n, 1-q) + (1-\pi)F(\frac{n-1}{2}, n, q)]$$

for $n = 1$. From Proposition 3 implies that $\mu_i^n(0, \emptyset) < \mu_i^1(0, \emptyset)$ for any $n \geq 3$.

A $(H, 1)$ executive in a close race earns an equilibrium payoff of ρ. If he chooses $y = 0$, his payoff is $(1-\rho)$. He is better off in equilibrium if $\rho \geq \frac{1}{2}$.

A $(L, 1)$ executive in a close race earns an equilibrium payoff of $\frac{\rho q(1-\pi)}{(1-q)\pi + q(1-\pi)}$. If he chooses $y = 0$ instead, he earns a payoff of $\frac{(1-q)\pi + (1-\rho)q(1-\pi)}{(1-q)\pi + q(1-\pi)}$. His equilibrium payoff exceeds this if and only if $\rho \geq \frac{\pi(1-q) + (1-\pi)q}{2(1-\pi)q} = \bar{\rho}$. Therefore the critical probability of state revelation sufficient to prevent pandering for the executive is the same as in any legislature. □

Proposition 4 *If $n-3$ or fewer new members are added to a group that serves the public interest, an equilibrium exists in which the new group chooses the correct policy with the same probability as the original group.*

Proof of Proposition 4:

Let K_n denote a set on n challengers, k_i. Assume that a PIE exists and that $\rho < \bar{\rho}$. Lemma 6 implies that for all $k_i \in K_n$, $k_i \notin (\mu_i^n(1,\emptyset), \mu_i^n(0,\emptyset)]$. Now hold ρ and all other parameters constant and add $l \leq n-3$ members to the legislature. The set of challengers is now $K_n \cup K_l$ where K_l denotes a set of l challengers. Let N denote the set of original members and let L denote the set of new members. In the $(n+l)$-member legislature, let m^n denote the set of messages sent by original members and let m^l denote the messages sent by new members. Define ψ^n and ψ^l similarly. Let $X(\psi^n)$ denote the set of all $\psi \in \Psi$ with ψ^n where Ψ is the type space for the $(n+l)-$member legislature. I now define an equilibrium for the $(n+l)-$member legislature:

Definition 2 (Expansion equilibrium) *Original members truthfully report their type*

$$\tilde{m}_i^*(\theta_i, s_i) = (\theta_i, s_i) \quad \textit{for all} \quad i \in N$$

New members all report the same type regardless of their type

$$\tilde{m}_i^*(\theta_i, s_i) = (L, 1)) \quad \textit{for all} \quad i \in L$$

Original members vote for the policy they believe is optimal

$$\tilde{v}_i^*(\theta_i, s_i, m) = 0 \quad \textit{if and only if} \quad \zeta_i(\theta_i, s_i, m) \geq 1/2 \quad \textit{for all} \quad i \in N$$

New members vote for the popular policy

$$\tilde{v}_i^*(\theta_i, s_i, m) = 0 \quad \textit{for all} \quad i \in L$$

Voter beliefs at each information set for original members are the same in the $(n+l)$ *legislature as in the* n*-member legislature.*

$$\mu_i^{n+l}(\phi) = \mu_i^n(\phi) \quad \textit{for all} \quad \phi \in \Phi \quad \textit{for all} \quad i \in N$$

Voter beliefs at each information set for new members equal their prior

$$\mu_i^{n+l}(\phi) = 1/2 \quad \textit{for all} \quad \phi \in \Phi \quad \textit{for all} \quad i \in L$$

Original member beliefs about the state are only informed by the messages of original members

For all $i \in N$,

$$\eta_i(X(\psi^n)) = \begin{cases} 1 & \textit{if} \quad \psi_{-i}^n = m_{-i}^n \quad \textit{and} \quad \psi_i = (\theta_i, m_i) \\ 0 & \textit{otherwise} \end{cases}$$

For all $i \in L$, $\eta_i(\psi)$ *satisfies Bayes' rule wherever possible.*

Off path, all members believe that the state is $\omega = 1$ *if at least one original member sends* $(H, 1)$ *and at least one other sends* $(H, 0)$.

For any $m^n \in \Psi^C$, $\eta_i(X(\psi^n)) > 0$ *if and only if* $\psi_j = (H, 1)$ *for all legislators with* $m_j = (H, 1)$.

I now show that if a PIE exists for an n-member legislature, the expansion equilibrium exists for the $(n+l)$-member legislature.

To do so, I first show that voter beliefs for new members satisfy Bayes rule. No new member has any influence on policy in equilibrium. Therefore for all $i \in L$ and all ϕ, $Pr(\phi|\theta_i = H) = Pr(\phi|\theta_i = L)$ which implies that $\mu_i(\phi) = 1/2$.

It is straightforward to check that $\mu_i^n(\phi) = \mu_i^{n+l}(\phi)$ for all $i \in N$. In equilibrium $Pr(y = \omega|\theta_i = H) = 1$ as in a PIE. Also as in a PIE, $Pr(y = \omega|\theta_i = L) = (1 - \frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}F^R$.

If an original member is a low type, the correct policy is chosen if any of the other original members are high ability or if all other members are low ability and $\frac{n+1}{2}$ or more correct signals are observed.

In the voting stage, all n original members vote as a bloc. No member of the legislature is pivotal if $n > \frac{n+l+1}{2}$. The right-hand side is maximized at $l = n-3$. For $l = n-3$, this inequality becomes $n > \frac{n-1}{2}$ which is true. Therefore no member can affect his payoff by affecting policy in the voting stage.

Given original members' beliefs, new messages have no affect on the legislature's policy choice. They therefore have no incentive to send a message other than the message proscribed by equilibrium.

Finally, because $k_i \notin (\mu_i^n(1,\emptyset), \mu_i^n(0,\emptyset)] = (\mu_i^{n+l}(1,\emptyset), \mu_i^{n+l}(0,\emptyset)]$ for all $k_i \in K_n$, no original member can benefit electorally from the popular policy when the unpopular policy should be chosen. Therefore original members cannot gain by deviating in the communication stage. □

Proposition 5 *If $\rho \geq \bar{\rho}$, a PIE exists.*

Proof of Proposition 5:

Lemma 6 immediately implies Proposition 5. □

Proposition 6 *If low ability legislators are privately uninformed ($q \leq \pi$), the unconventional policy, $y = 1$, is more popular: $\mu_i(0) < \mu_i(1)$. If low ability legislators are privately informed ($q > \pi$)), the conventional policy, $y = 0$, is more popular: $\mu_i(1) < \mu_i(0)$.*

Proof of Proposition 6: Lemma 5 implies Proposition 6. □

Proposition 7 *As low ability legislators become more informed, fewer individual races are close.*

Proof of Proposition 7:

Lemma 4 part 3 immediately implies Proposition 7.

Proposition 8 *With privately uninformed legislators, fewer individual races are close in larger groups than smaller groups.*

Proof of Proposition 8:

Lemma 4 implies Proposition 8. □

5 Lemma Proofs

Proof of Lemma 1:

$n_0(q)$ is the smallest n_0 such that

$$\frac{Pr(n_0|y=0)Pr(y=0)}{Pr(n_0|y=0)Pr(y=0)+Pr(n_0|y=1)Pr(y=1)} \geq 1/2$$

or

$$\frac{\pi q^{n_0}(1-q)^{n-n_0}}{\pi q^{n_0}(1-q)^{n-n_0}+(1-\pi)(1-q)^{n_0}q^{n-n_0}} \geq 1/2$$

This condition further reduces to

$$(\frac{q}{1-q})^{2n_0-n} \geq \frac{1-\pi}{\pi}$$

Exponentiation yields the condition

$$n_0 \geq \frac{\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi})+n}{2}$$

A group of low ability members therefore chooses $y=0$ given q if and only if

$$n_0 \geq n_0(q) \equiv \lceil \frac{\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi})+n}{2} \rceil$$

where $\lceil \cdot \rceil$ is the ceiling function which returns the smallest integer greater than or equal to its argument.

Note that $q/(1-q) > 1$. Thus $\log_{\frac{q}{1-q}}(x) < 0$ is negative for $x < 1$, positive for $x > 1$, continuous, and strictly increasing on $\mathbb{R}_+$. For an arbitrary base a and a fixed argument x,

$$\frac{\partial}{\partial a}\log_a x = \frac{-\ln x}{(\ln a)^2 a}$$

Note that $\frac{q}{1-q}$ is strictly positive and increasing in q and that $\frac{1-\pi}{\pi} < 1$. Therefore for an arbitrary $\pi \in (1/2, 1)$, $\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi})$ is strictly increasing and continuous in q. This implies that

$$\frac{\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi}) + n}{2}$$

is strictly increasing and continuous in q. Strictly increasing implies that $n_0(q)$ is weakly increasing in q by the properties of the ceiling function. This establishes that $n_0(q)$ is increasing in q. Continuity will imply that $n_0(q)$ is onto N_0 after it is established that $n_0(1/2) = 0$ and $n_0(q) = \frac{n+1}{2}$ for $q > \pi$.

It is straightforward to check that $n_0(1/2) = 0$ and $n_0(\pi) = \frac{n-1}{2}$ by simply substituting appropriate values into $n_0(q)$. To establish that $n_0(q) = \frac{n+1}{2}$ for all $q > \pi$, it is sufficient to show that

$$\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi}) \in (-1, 1]$$

Because $\frac{1-\pi}{\pi} < 1$, $\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi}) < 0$. This exceeds -1 if

$$\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi}) > -1$$

The inequality simplifies to $\frac{1-\pi}{\pi} > \frac{1-q}{q}$ which is true if $q > \pi$. Therefore $n_0(q) = \frac{n+1}{2}$ for all $q > \pi$. The continuity of $\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi})$ in q now implies that $n_0(q)$ is onto N_0. $\square$

Proof of Lemma 2:

Proof that $\lambda(1) = \pi$ and $\lambda(1/2) = 1$:

It follows immediately from Lemma 1 and Equation 1 that $\lambda(1/2) = 1$. Now note that if $q = 1$, then low types always make the correct decision, just like high types. Therefore

$\lambda(1) = \pi$.

Proof that $\lambda(q)$ is strictly increasing and continuous on $(\pi, 1]$:

$\lambda(q)$ depends on the group's decision rule, $n_0(q)$, and the direct effect of q on the quality of information for a given threshold. To analyze the direct effect of q, let

$$\nu(q; \tilde{n}_0) \equiv \pi(1 - F(\tilde{n}_0 - 1, n, q)) + (1 - \pi)(1 - F(\tilde{n}_0 - 1, n, 1 - q))$$

The function $\nu(q; \tilde{n}_0)$ takes an exogenous decision rule, $\tilde{n}_0 \in N_0$, as a parameter and returns the probability that $n_0 \geq \tilde{n}_0$ as a function of q. It follows from the properties of the binomial cdf in success probability that for all $\tilde{n}_0 \in N_0$, $\nu(q; \tilde{n})$ is continuously differentiable in q on $(1/2, 1)$. Its derivative is

$$\frac{\partial}{\partial q}\nu(q; \tilde{n}_0) = n\binom{n-1}{\tilde{n}_0 - 1}[q^{\tilde{n}_0 - 1}(1-q)^{n-\tilde{n}_0}\pi - (1-q)^{\tilde{n}_0 - 1}q^{n-\tilde{n}_0}(1-\pi)]$$

Examining this reveals that

$$\text{sign}(\frac{\partial \nu(q; \tilde{n}_0)}{\partial q}) = \text{sign}(\frac{\pi}{1-\pi} - (\frac{1-q}{q})^{2\tilde{n}_0 - 1 - n}) \tag{4}$$

From Lemma 1, $n_0(q) = \frac{n+1}{2}$ for all $q \in (\pi, 1)$. Therefore $\lambda(q) = \nu(q; \tilde{n}_0)$ for all $q \in (\pi, 1)$. For $\tilde{n}_0 = \frac{n+1}{2}$ Equation 4 implies that $\nu(q, \frac{n+1}{2})$ is strictly increasing on $Q(\frac{n+1}{2}) = (1/2, 1)$ if $\frac{\pi}{1-\pi} - 1 > 0$. The inequality is satisfied because $\pi > 1/2$. Therefore $\lambda(q)$ is strictly and continuously increasing on $(\pi, 1)$. Having characterized $\lambda(q)$ on $(1/2, 1)$, it is straightforward to extend this characterization to $(1/2, 1]$. As q approaches one, $F(\frac{n-1}{2}, n, q)$ approaches zero and $F(\frac{n-1}{2}, n, 1-q)$ approaches one. Therefore $\lim_{q \to 1} \nu(q; \frac{n+1}{2}) = \pi$. It was established above that $\lambda(q) = \pi$ at $q = 1$. Therefore $\lambda(q)$ is strictly increasing and continuous on $(\pi, 1]$.

Proof that $\lambda(q)$ is decreasing on $[1/2, \pi]$:

It will be useful to let $\tilde{n}_0 \in N_0$ represent a possible number of $s_i = 0$ signals in a group of n low ability members. To summarize my notation, n_0 is the number of $s_i = 0$ signals that is actually realized, $n_0(q)$ is the number of signals given q such that a group of low ability members chooses $y = 0$ if and only if $n_0 \geq n_0(q)$, and $\tilde{n}_0$ is a free variable used to make statements such as "if $n_0 > \tilde{n}_0$" or "assume $n_0(q) = \tilde{n}_0$."

It follows from Lemma 1 that for all $\tilde{n}_0 \in N_0$, an interval $Q(\tilde{n}_0) \subset [1/2, 1)$ exists such that $n_0(q) = \tilde{n}_0$ for all $q \in Q(\tilde{n}_0)$. Moreover, $\bigcup_{\tilde{n}_0 \in N_0} Q(\tilde{n}_0) = [1/2, 1)$, $\bigcap_{\tilde{n}_0 \in N_0} Q(\tilde{n}_0) = \emptyset$, and $\sup Q(\tilde{n}_0) = \inf Q(\tilde{n}_0 + 1)$. The bounds on each set $Q(n_0)$ are determined by the values of q where $n_0(q)$ discontinuously rises. The maximum of each interval is found by inverting $n_0(\cdot)$,

$$q(\tilde{n}_0) \equiv \frac{1}{1 + (\frac{1-\pi}{\pi})^{\frac{1}{n-2\tilde{n}_0}}}$$

For each $\tilde{n}_0 \in N_0 \setminus \frac{n+1}{2}$, $q(\tilde{n}_0)$ is the largest value of q such that $n_0(q) = \tilde{n}_0$. The interval $Q(0)$ can now be expressed as $Q(0) = [0, q(0)]$. Similarly, $Q(1) = (q(0), q(1)]$. This generalizes such that for any $\tilde{n}_0 \in N_0 \setminus \{\frac{n+1}{2}, 0\}$, $Q(\tilde{n}_0) = (q(\tilde{n}_0 - 1), q(\tilde{n}_0)]$. These intervals allow $\lambda(q)$ to be expressed as a discontinuous piecewise function of q on $[1/2, \pi]$,

$$\lambda(q) = \nu(q; \tilde{n}_0) \quad \text{if} \quad q \in Q(\tilde{n}_0)$$

I first show that $\lambda(q)$ discontinuous at each $q(\tilde{n}_0)$ and that $\lambda(q)$ decreases at each discontinuity. I then show that $\lambda(q)$ is continuously decreasing on each $Q(\tilde{n}_0)$.

It was shown above that each $\nu(q; \tilde{n}_0)$ is continuous and differentiable in q. Therefore $\lambda(q)$ is continuous in q on each $Q(\tilde{n}_0)$. Note that for a fixed q, $\nu(q; \tilde{n}_0)$ is decreasing in $\tilde{n}_0$. This follows simply from the properties of a cdf. Therefore, for each $q(\tilde{n}_0)$ where $n_0(q(\tilde{n}_0)) = \tilde{n}_0$ and $n_0(q(\tilde{n}_0) + \epsilon) = \tilde{n}_0 + 1$ for $\epsilon > 0$ arbitrarily small, $\lambda(q)$ discontinuously falls:

$$\nu(q(\tilde{n}_0); \tilde{n}_0) > \lim_{q \to q^+(\tilde{n}_0)} \nu(q; \tilde{n}_0 + 1) = \nu(q(\tilde{n}_0); \tilde{n}_0 + 1)$$

Now I show that $\lambda(q)$ is continuously decreasing on $Q(\tilde{n}_0)$ for all $\tilde{n}_0 \in \{0, 1, ... \frac{n-1}{2}\}$. First consider $Q(0)$. On this interval, the group selects $y = 0$ if $n_0 \geq 0$. This event occurs with probability one i.e. $\nu(q,0) = 1$. Thus for $q \in Q(0) = [1/2, q(0)]$, $\lambda(q) = 1$. Now consider $\tilde{n}_0 \in \{1, ...\frac{n-1}{2}\}$. It follows from Equation (4) that for each $\nu(q;\tilde{n}_0)$ for $\tilde{n}_0 \in \{1, ...\frac{n-1}{2}\}$, there is a unique $\tilde{q}(\tilde{n}_0)$ such that

$$\frac{\partial \nu(q;\tilde{n}_0)}{\partial q}\bigg|_{\tilde{q}(\tilde{n}_0)} = 0$$

If $q < \tilde{q}(n_0)$, then $\nu(q;n_0)$ is increasing. If $q > \tilde{q}(\tilde{n}_0)$, then $\nu(q;\tilde{n}_0)$ is decreasing. To see this, recall Equation 4 and note that $\frac{\pi}{1-\pi}$ is constant. The term $(\frac{1-q}{q})^{2\tilde{n}_0-1-n}$ equals 1 at $q = 1/2$ and is strictly increasing on $[0,1]$ with $\lim_{q\to 1^-}(\frac{1-q}{q})^{2\tilde{n}_0-1-n} = \infty$. For a given $\tilde{n}_0$ then,

$$\frac{\tilde{q}(\tilde{n}_0)}{1-\tilde{q}(\tilde{n}_0)} = \frac{\pi}{1-\pi}^{\frac{1}{n+1-2\tilde{n}_0}}$$

The proof proceeds by showing that for all $\tilde{n}_0 \in \{1, ...\frac{n-1}{2}\}$, $\tilde{q}(\tilde{n}_0) < q(\tilde{n}_0 - 1)$. This implies that on each $Q(\tilde{n}_0)$, the derivative of $\nu(q;\tilde{n}_0)$ is negative. It will be helpful to note that for any $\tilde{n}_0 \in \{1, ...\frac{n-1}{2}\}$, $\tilde{n}_0$ can be expressed as $\frac{n-a}{2}$ for $a \in \{1, 3, ..., n-4, n-2\}$. With this notation, $n_0(q) < \tilde{n}_0$ if

$$\log_{\frac{q}{1-q}}(\frac{1-\pi}{\pi}) \leq -a-2$$

Now for any $\tilde{n}_0$ in this range and its corresponding a term, $\tilde{q}(\tilde{n}_0) = \tilde{q}(\frac{n-a}{2}) = (\frac{\pi}{1-\pi})^{\frac{1}{a+1}}$. It follows that $n_0(\tilde{q}(\tilde{n}_0)) < \tilde{n}_0$ if

$$\log_{(\frac{\pi}{1-\pi})^{\frac{1}{a+1}}}(\frac{1-\pi}{\pi}) \leq -a-2$$

$$(\frac{\pi}{1-\pi})^{\frac{1}{a+1}} \leq (\frac{\pi}{1-\pi})^{a+2}$$

$$(1-\pi) \leq \pi$$

which is true because $\pi > 1-\pi$. Thus for all $\tilde{n}_0$, $\tilde{q}(\tilde{n}_0) < q(\tilde{n}_0 - 1)$ i.e. for all $q \in Q(\tilde{n}_0)$, $q > \tilde{q}(\tilde{n}_0)$ and thus $\nu(q,\tilde{n}_0)$ is strictly decreasing on $Q(\tilde{n}_0)$ for $\tilde{n}_0 \in \{1, ...\frac{n-1}{2}\}$.

Altogether then, I have shown that $\lambda(q)$ is constant on $[1/2, q(0)]$, strictly and continuously decreasing on all $(q(\tilde{n}_0 - 1), q(\tilde{n}_0)]$ for $\tilde{n}_0 \in \{2, ... \frac{n-1}{2}\}$, and $\lambda(q(\tilde{n}_0 - 1)) > \lim_{q \downarrow q(\tilde{n}_0)}$ for $\tilde{n}_0 \in \{1, 2, ... \frac{n-1}{2}\}$. Therefore $\lambda(q)$ is decreasing on $[1/2, \pi]$.

Proof that $\lambda(\pi) > \pi$:

It was just established that $\nu(q, \frac{n-1}{2})$ is strictly decreasing in q for $q \geq q(\frac{n-3}{2})$. Now note that $\nu(1, \frac{n-1}{2}) = \pi$:

$$\nu(1, \frac{n-1}{2}) = \pi(1 - F(\frac{n-3}{2}, n, 1)) + (1-\pi)(1 - F(\frac{n-3}{2}, n, 0)) = \pi$$

Therefore $\nu(\pi, \frac{n-1}{2}) > \pi$. Because $\lambda(q) = \nu(q, \frac{n-1}{2})$ on $(q(\frac{n-3}{2}), \pi]$ and $\pi < 1$, $\lambda(\pi) > \pi$.

Proof that $\lambda(\pi) - \pi > \pi - \lim_{q \to^+ \pi} \lambda(q)$:

Substitution and algebra allows the inequality to be rewritten

$$1 < \sum_{i=\frac{n-1}{2}}^{n} \binom{n}{i} \pi^i (1-\pi)^{n-i} + \sum_{i=\frac{n+1}{2}}^{n} \binom{n}{i} (1-\pi)^{i+1} \pi^{n-i-1}$$

Note that the expression on the right-hand-side (RHS) of the inequality equals 1 at $\pi = 1$. The derivative of the RHS with respect to π reveals that it is strictly decreasing in π on $[1/2, 1]$. This implies that the RHS is greater than 1 for $\pi < 1$. The derivative of the RHS with respect to π is

$$(\frac{n(1-2\pi)-1}{\pi(1-\pi)}) \sum_{i=\frac{n+1}{2}}^{n} \binom{n}{i} [\pi^i (1-\pi)^{n-i} + (1-\pi)^{i+1} \pi^{n-i-1}] + (\frac{n(1-2\pi)-1}{2\pi(1-\pi)}) \binom{n}{\frac{n-1}{2}} \pi^{\frac{n-1}{2}} (1-\pi)^{\frac{n+1}{2}}$$

Because $\pi \geq 1/2$, $(\frac{n(1-2\pi)-1}{\pi(1-\pi)})$ and $(\frac{n(1-2\pi)-1}{2\pi(1-\pi)})$ are negative. Therefore the derivative is negative. Therefore the inequality is satisfied for all $\pi < 1$. □

Lemma 7 is useful for proving Lemma 3

Lemma 7 *Let* $b \in \{1, 2, , , , \}$ *and* $a \in \{0, 1, , , , b-1\}$.

$$\pi \sum_{i=b+1-a}^{2b+1} \binom{2b+1}{i} q^i (1-q)^{2b+1-i} + (1-\pi) \sum_{i=b+1-a}^{2b+1} \binom{2b+1}{i} (1-q)^i q^{2b+1-i}$$

is increasing in b for $a = 0$ and decreasing in b otherwise.

Proof of Lemma 7:

I first show the result for $a = 0$. For $a = 0$, the term in the Lemma becomes

$$\pi \sum_{i=b+1-a}^{2b+1} \binom{2b+1}{i} q^i (1-q)^{2b+1-i} + (1-\pi)(1 - \sum_{i=b+1-a}^{2b+1} \binom{2b+1}{i} q^i (1-q)^{2b+1-i})$$

Because $\pi > 1 - \pi$, the term is increasing in b if $\sum_{i=\frac{n+1}{2}}^{n} q^i (1-q)^{n-i}$ is increasing in b. Ben-Yashar and Paroush (2000) show the following. For any $b \in \mathbb{N}$,

$$\sum_{i=b+1}^{2b+1} \binom{2b+1}{i} q^i (1-q)^{2b+1-i} = q^2 A(q) + (1-(1-q)^2) B(q) + C(q)$$

where

$$A(q) = \binom{2b-1}{b-1} q^{b-1} (1-q)^b$$

$$B(q) = \binom{2b-1}{b} q^b (1-q)^{b-1}$$

$$C(q) = \sum_{i=b+1}^{2b-1} \binom{2b-1}{i} q^i (1-q)^{2b-1-i}$$

The first term expresses the probability that a group of $n = 2b+1$ low ability types receives at least $(n+1)/2$ correct signals. The decomposition isolates two members from this group where the probability that a group of $n-2 = 2b-1$ low ability members receives at least $(n-1)/2$ correct signals is $B(q) + C(q)$. The larger group selects the correct policy if and only if

$$q^2 A(q) - (1-q)^2 B(q) > 0$$

This condition simplifies to $q > (1-q)$ for all b which is true by assumption. Therefore

$\sum_{i=\frac{n+1}{2}}^{n} \binom{n}{i} q^i (1-q)^{n-i}$ is monotonically increasing in odd n.

I now extend their result to $a > 0$. Because $\pi > 1/2$, it is sufficient to show that

$$\sum_{i=b-a}^{2b-1} \binom{2b-1}{i} q^i (1-q)^{2b-1-i} - \sum_{i=b+1-a}^{2b+1} \binom{2b+1}{i} q^i (1-q)^{2b+1-i}$$
$$>$$
$$\sum_{i=b+1-a}^{2b+1} \binom{2b+1}{i} (1-q)^i q^{2b+1-i} - \sum_{i=b-a}^{2b-1} \binom{2b-1}{i} (1-q)^i q^{2b-1-i}$$

To prove this, note that

$$\sum_{i=b+1-a}^{2b+1} \binom{2b+1}{i} q^i (1-q)^{2b+1-i} = q^2 A(q) + (1-(1-q)^2) B(q) + C(q)$$

where

$$A(q) = \binom{2b-1}{b-a-1} q^{b-a-1} (1-q)^{b+a}$$

$$B(q) = \binom{2b-1}{b-a} q^{b-a} (1-q)^{b-1+a}$$

$$C(q) = \sum_{i=b+1-a}^{2b-1} \binom{2b-1}{i} q^i (1-q)^{2b-1-i}$$

The inequality can thus be expressed

$$(1-q)^2 B(q) - q^2 A(q) > (1-q)^2 A(1-q) - q^2 B(1-q)$$

Substituting and rearranging yields

$$\frac{(1-q) + q(\frac{q}{1-q})^{2a}}{(b-1+a)!(b-a)!} > \frac{q + (1-q)(\frac{q}{1-q})^{2a}}{(b-a-1)!(b+a)!}$$

Because $q > 1/2$, for $a \geq 1$, the numerator on the LHS exceeds that on the RHS. (Note that for $a = 0$ they are equivalent.) The denominator on the LHS and RHS can be expressed respectively as

$$\prod_{i=1-a}^{a} (b + i - 1)$$

and

$$\prod_{i=1-a}^{a} (b + i)$$

Thus for $a \geq 1$, the denominator on the RHS is larger. (Note that for $a = 0$, they are equivalent: $0! = 1$). This establishes that the sufficient condition holds for an arbitrary b for all a. $\square$

I now prove Lemma 3.

Proof of Lemma 3:

For any odd n, $\lambda(q)$ can be expressed as the term in Lemma 7 for $a = \frac{n+1}{2} - n_0(q)$ and $b = \frac{n-1}{2}$. For $q > \pi$, $n_0(q) = \frac{n+1}{2}$ by Lemma 1. Therefore $a = 0$ and $\lambda(q)$ is increasing in n by Lemma 7 if $q > \pi$. For $q \leq \pi$, $n_0(q) < \frac{n+1}{2}$ by Lemma 1. Therefore $a \geq 1$ and $\lambda(q)$ is decreasing in n. $\square$

Proof of Lemma 4: Note that $\mu_i(0, \emptyset)$ is continuously decreasing in $\lambda(q)$ and $\mu_i(1, \emptyset)$ is continuously increasing in $\lambda(q)$. Also note that $\mu_i(0, \emptyset) = \mu_i(1, \emptyset) = 1/2$ if $\lambda(q) = \pi$. Further, $\mu_i(0, \emptyset) > 1/2 > \mu_i(1, \emptyset)$ if and only if $\lambda(q) < \pi$. Therefore Lemma 2 implies the first two parts of Lemma 4.

To establish part 3, first note that Lemma 2 implies that $|\mu_i(0, \emptyset) - \mu_i(1, \emptyset)|$ is decreasing on $[1/2, \pi]$ and $(\pi, 1]$. All that remains is to show that the distance is also decreasing on $(\pi - \epsilon, \pi + \epsilon)$. Let $\lambda'(\pi) \equiv \lim_{q \to \pi^+} \lambda(q)$. For $q = \pi$,

$$|\mu_i(0, \emptyset) - \mu_i(1, \emptyset)| = \frac{1 - \pi}{(1 - \pi)(2 - \frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(1 - \lambda(\pi))} - \frac{\pi}{\pi(2 - \frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}\lambda(\pi)}$$

For $q = \pi + \epsilon$,

$$|\mu_i(0,\emptyset) - \mu_i(1,\emptyset)| = \frac{\pi}{\pi(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}\lambda'(\pi)} - \frac{1-\pi}{(1-\pi)(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(1-\lambda'(\pi))}$$

From Lemma 2, $\lambda(\pi) > \pi$ and $\pi > \lambda'(\pi)$. Thus $\lambda'(\pi) = \pi - a$ and $\lambda(\pi) = \pi + b$ for $a, b \in (0, 1-\pi)$. Note that

$$\frac{1-\pi}{(1-\pi)(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(1-\pi-b)} - \frac{\pi}{\pi(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(\pi+b)}$$

is strictly increasing in b. Now let $b = a$ and compare the two differences. For $b = a$,

$$\frac{\pi}{\pi(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(\pi-a)} - \frac{1-\pi}{(1-\pi)(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(1-\pi+a)}$$

$$<$$

$$\frac{1-\pi}{(1-\pi)(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(1-\pi-a)} - \frac{\pi}{\pi(2-\frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}}(\pi+a)}$$

if $2\pi a - a > 0$. Because $a > 0$ and $\pi > 1 - \pi$, this inequality holds. Thus if $b \geq a$, $|\mu_i(0,\emptyset) - \mu_i(1,\emptyset)|$ is larger for $q = \pi$ than $q = \pi + \epsilon$. Rearrange terms to get $b = \lambda(\pi) - \pi$ and $a = \pi - \lambda'(\pi)$. From Lemma 2, $b > a$. This establishes the last part of Lemma 4. $\square$

Proof of Lemma 5:

From (2), $\mu_i(0,\emptyset)$ is decreasing in n if

$$(1 - \frac{1}{2^{n-1}})\pi + (\frac{1}{2^{n-1}})\lambda(q)$$

is increasing in n.

From (3), $\mu_i(1,\emptyset)$ is increasing in n if

$$(1 - \frac{1}{2^{n-1}})(1-\pi) + (\frac{1}{2^{n-1}})(1-\lambda(q))$$

is decreasing in n.

For $q \leq \pi$, Lemma 2 implies that $\lambda(q) > \pi$ and that $\lambda(q)$ is decreasing in n. Therefore $\mu_i(0, \emptyset)$ is increasing in n and $\mu_i(1, \emptyset)$ decreasing in n for $q \leq \pi$. For $q > \pi$, Lemma 2 implies that $\lambda(q) < \pi$ and that $\lambda(q)$ is increasing in n. Therefore $\mu_i(0, \emptyset)$ is decreasing in n and $\mu_i(1, \emptyset)$ increasing in n for $q > \pi$. □

Proof of Lemma 6: I first characterize voter beliefs when the state is revealed. For all $q < 1$, $y \neq \omega$ only if all members are low ability. Therefore $\mu_i(0, 1) = \mu_i(1, 0) = 0$. By Lemma 1, a group of low ability members selects $y = 0$ if and only if $n_0 \geq \frac{n+1}{2}$. Conditional on $\omega = 0$ then, the group selects $y = 0$ with probability $1 - F(\frac{n-1}{2}, n, q)$. Conditional on $\omega = 1$, the group selects $y = 1$ with probability $F(\frac{n-1}{2}, n, 1-q)$. For n odd, $1-F(\frac{n-1}{2}, n, q) = F(\frac{n-1}{2}, n, 1-q)$. Therefore

$$\mu_i(0,0) = \mu_i(1,1) = \frac{1}{1 + (1 - \frac{1}{2^{n-1}}) + \frac{1}{2^{n-1}} F(\frac{n-1}{2}, n, 1-q)} > 0$$

Because $0 = \mu_i(1, 0) = \mu_i(0, 1) < \mu_i(1, 1) = \mu_i(0, 0)$, all legislators prefer to match policy to the state if the state is revealed. If policy does not match the state, all legislators lose reelection (unless $k_i = 0$). If the state is not revealed, then legislators with challengers $k_i \in (\mu_i(1, \emptyset), \mu_i(0, \emptyset)]$ win reelection if and only if $y = 0$. Legislators with challengers outside of this interval either win the election regardless of policy ($k_i \leq \mu_i(1, \emptyset)$) or lose ($k_i > \mu_i(0, \emptyset)$). These legislators can do no better than to maximize the probability that policy matches the state. The legislators in close races are thus the only legislators who may want to deviate from equilibrium strategies in order to raise the probability that $y = 0$ is chosen over $y = 1$. Because legislators vote unanimously in the voting stage in equilibrium, no legislator is pivotal in the voting stage. Any profitable deviations from equilibrium must therefore come in the communication stage.

There are four types of legislators and each type can lie in three ways. To rule out some possible deviations completely, first note that because the probability of state matching is maximized in equilibrium, all lies reduce the probability that $y = \omega$. That is, a lie must raise the probability that $y = 0$ relative to equilibrium. An immediate implication of this is that the $(H, 0)$ type can never gain by lying. In equilibrium he is reelected with probability one. A second implication is that no member can gain from telling a lie that raises the probability that $y = 1$ is selected. This rules out lies by the $(L, 0)$ and $(L, 1)$ type that their type is $(H, 1)$. It also rules out a lie by the $(L, 0)$ type that he is a $(L, 1)$ type.

This leaves the following lies to consider. The $(H, 1)$ type may misreport $m_i \in \{(H, 0), (L, 0), (L, 1)\}$. The $(L, 1)$ type may misreport $m_i \in \{(H, 0), (L, 0)\}$. The $(L, 0)$ type may misreport $m_i = (H, 0)$.

I begin with the $(H, 1)$ type. Lies are only effective if no other member is a high type. His equilibrium payoff in this event is ρ. If he sends the message $(L, 1)$, he earns an expected payoff of

$$F(\frac{n-1}{2}, n-1, 1-q)\rho + (1 - F(\frac{n-1}{2}, n-1, 1-q))(1-\rho)$$

If he sends $(L, 0)$, he earns

$$F(\frac{n-3}{2}, n-1, 1-q)\rho + (1 - F(\frac{n-3}{2}, n-1, 1-q))(1-\rho)$$

Finally, if he sends $(H, 0)$, he earns an payoff of $(1 - \rho)$. In all three cases, the lie is unprofitable if and only if $\rho \geq \frac{1}{2}$.

Now consider the $(L, 1)$ type and the critical value of ρ that prevents him from profiting from sending $m_i = (L, 0)$.

Let $\gamma(0) \equiv Pr(\omega = 0|\theta_i = L, s_i = 0)$ and $\gamma(1) \equiv Pr(\omega = 0|\theta_i = L, s_i = 1)$. In the event that no other member is a high type, the $(L, 1)$ type earns an equilibrium payoff of

$$\gamma(1)(1 - F(\frac{n-1}{2}, n-1, q)) + (1-\gamma(1))[F(\frac{n-1}{2}, n-1, 1-q)\rho + (1 - F(\frac{n-1}{2}, n-1, 1-q))(1-\rho)]$$

If the type $(L, 1)$ legislator sends message $m_i = (L, 0)$, his expected payoff if no other member is a high type is

$$\gamma(1)(1-F(\frac{n-3}{2},n-1,q))+(1-\gamma(1))[(F(\frac{n-3}{2},n-1,1-q)\rho+(1-F(\frac{n-3}{2},n-1,1-q))(1-\rho))]$$

This is less than or equal to his equilibrium payoff if and only if

$$\rho \geq \bar{\rho} \equiv \frac{1}{2(1-\gamma(1))}$$

First note that $\bar{\rho} \geq \frac{1}{2}$. Thus if the $(L,1)$ type cannot profitably misreport $m_i = (L,0)$, no lie by the $(H,1)$ type is profitable. If the type $(L,1)$ legislator sends $m_i = (H,0)$, his expected payoff if no member is a high type is

$$\gamma(1) + (1-\gamma(1))(1-\rho)$$

This exceeds his equilibrium payoff if and only if

$$\rho \geq \rho_1 \equiv \frac{\gamma(1)F(\frac{n-1}{2},n-1,q) + (1-\gamma(1))F(\frac{n-1}{2},n-1,1-q)}{2(1-\gamma(1))F(\frac{n-1}{2},n-1,1-q)}$$

Comparing and simplifying reveals that $\bar{\rho} \geq \rho_1$ if

$$F(\frac{n-1}{2},n-1,1-q) \geq F(\frac{n-1}{2},n-1,q)$$

Because $q > 1/2$, this inequality is true. Thus if the $(L,1)$ type cannot profitably misreport $m_i = (L,0)$, then it is also unprofitable for the $(L,1)$ type to misreport $m_i = (H,0)$.

There is now only one remaining lie to consider, the $(L,0)$ type misreporting $m_i = (H,0)$. A $(L,0)$ type earns an equilibrium payoff if no member is a high type of

$$\gamma(0)(1-F(\frac{n-3}{2},n-1,q))+(1-\gamma(0))[F(\frac{n-3}{2},n-1,1-q)\rho+(1-F(\frac{n-3}{2},n-1,1-q))(1-\rho))]$$

His payoff if he sends message $m_i = (H, 0)$ is

$$\gamma(0) + (1 - \gamma(0))(1 - \rho)$$

He is better off in equilibrium if and only if

$$\rho \geq \rho_0 \equiv \frac{\gamma(0)F(\frac{n-3}{2}, n-1, q) + (1-\gamma(0))F(\frac{n-3}{2}, n-1, 1-q)}{2(1-\gamma(0))F(\frac{n-3}{2}, n-1, 1-q)}$$

The following two lemmas will be useful for showing that $\bar{\rho} \geq \rho_0$.

Lemma 8 *For $q \in (1/2, 1)$ and n even,*

$$\frac{F(\frac{n}{2}, n, 1-q)}{F(\frac{n}{2}, n, q)} < \frac{F(\frac{n+2}{2}, n+2, 1-q)}{F(\frac{n+2}{2}, n+2, q)}$$

Proof of Lemma 8: Ben-Yashar and Paroush (2000) provide the following identity:

$$\sum_{i=b+1}^{2b} \binom{2b}{i} q^i (1-q)^{2b-i} = q^2 A(q) + (1 - (1-q)^2) B(q) + C(q)$$

where

$$A(q) = \binom{2b-2}{b-1} q^{b-1} (1-q)^{b-1}$$

$$B(q) = \binom{2b-2}{b} q^b (1-q)^{b-2}$$

$$C(q) = \sum_{i=b+1}^{2b-2} \binom{2b-2}{i} q^i (1-q)^{2b-2-i}$$

The inequality in the Lemma can therefore be expressed

$$\frac{1-B(1-q)-C(1-q)}{1-B(q)-C(q)} < \frac{1-[(1-q)^2A(1-q)-(1-q^2)B(1-q)-C(1-q)]}{1-[q^2A(q)-(1-(1-q)^2)B(q)-C(q)]}$$

This condition simplifies to

$$\binom{2b-2}{b-1}q^{b-1}(1-q)^{b+1}(1-C(q))+\binom{2b-2}{b}^2 q^{2b}(1-q)^{2b-2} <$$

$$\binom{2b-2}{b-1}q^{b+1}(1-q)^{b-1}(1-C(1-q))+\binom{2b-2}{b}^2 q^{2b-2}(1-q)^{2b}$$

which further simplifies to

$$0 < (\binom{2b-2}{b-1}^2-\binom{2b-2}{b}^2)[q^{b+1}(1-q)^{b-1}-(1-q)^{b+1}q^{b-1}]$$

$$+\binom{2b-2}{b-1}\sum_{i=0}^{b-2}\binom{2b-2}{i}[(1-q)^i q^{2b-i}-q^i(1-q)^{2b-i}]$$

Note that $\frac{2b-2}{2}=b-1$. Therefore $\binom{2b-2}{b-1}>\binom{2b-2}{b}$. The term $q^{b+1}(1-q)^{b-1}-(1-q)^{b+1}q^{b-1}$ is positive if $(\frac{q}{1-q})^2>1$ which is true because $q>1/2$. Finally, for a given i, $[(1-q)^i q^{2b-i}-q^i(1-q)^{2b-i}$ is positive if $(\frac{q}{1-q})^{2(b-i)}>1$. The maximum value that i can take is $b-2$. Therefore $q>1-q$ implies this term is positive for all i. □

Lemma 9 ρ_0 *is strictly decreasing in* n.

Proof of Lemma 9:

Let $\rho_0(n)$ denote the value of ρ_0 for an n-member legislature. For an arbitrary n, $\rho_0(n)>\rho_0(n+2)$ if

$$\frac{F(\frac{n-3}{2},n-1,1-q)}{F(\frac{n-3}{2},n-1,q)} < \frac{F(\frac{n-1}{2},n+1,1-q)}{F(\frac{n-1}{2},n+1,q)}$$

From Lemma 8 inequality is satisfied. □

I can now show that $\bar{\rho}\geq\rho_0$. From Lemma 9, ρ_0 is maximized at $n=3$. Note that $\bar{\rho}$ is constant in n. Therefore $\bar{\rho}\geq\rho_0$ for all n if the inequality is satisfied for $n=3$. For $n=3$,

$F(\frac{n-3}{2}, n-1, q) = (1-q)^2$ and $F(\frac{n-3}{2}, n-1, 1-q) = q^2$. Plugging π and q into $\gamma(0)$ and simplifying yields

$$\rho_0 = \frac{\pi(1-q) + (1-\pi)q}{2(1-\pi)q} = \bar{\rho}$$

Thus if $\rho \geq \bar{\rho}$, no type can gain by deviating from equilibrium. If $\rho < \bar{\rho}$, the $(L, 1)$ type is strictly better off reporting $m_i = (L, 0)$ and a PIE therefore does not exist. This establishes Lemma 6. □

References

Ben-Yashar, R. and J. Paroush (2000). A nonasymptotic Condorcet jury theorem. *Social Choice and Welfare 17*, 189–199.